Indian Statistical Institute, Bangalore
M. Math. First Year
Second Semester - Topology II

Mid-Semester Exam　　Duration: 3 hours　　Date : February 22, 2016

Max Marks: 100

Remark: There are five questions, each carrying 25 marks. Answer any four.

1. For $n \geq 0$, there is a covariant functor H_n from the category of topological spaces to the category of groups. For the m-sphere S^m and the $m - ball$ B^m, we have

$$H_n(S^m) = \begin{cases} \mathbb{Z} & if\ n = 0\ or\ m, \\ \{0\} & otherwise \end{cases}$$

$$H_n(B^m) = \begin{cases} \mathbb{Z} & if\ n = 0, \\ \{0\} & otherwise \end{cases}$$

   (a) Assuming these results, prove that for $m \geq 1, S^{m-1}$ is not a retract of B^m.

   (b) Assuming (a), prove that every continuous function from B^m to B^m has a fixed point.

2. (a) Let $\exp : \mathbb{R} \to S^1$ be defined by $\exp\ (t) = e^{2\ \pi\ i\ t}$. What do we mean by the statement that exp is "essentially" the quotient map from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$? Make it precise and prove it.

   (b) If H is a closed normal subgroup of a topological group G, then show that the quotient map from G to G/H is a covering projection iff H is discrete.

3. Let $X = \{(x, \sin \frac{1}{x}) : 0 < x \leq \frac{1}{2\pi}\} \cup \{(0, y) : -1 \leq y \leq 1\} \subseteq \mathbb{R}^2$. Find the path components of X, with proof. Show that $\pi_1(X, x)$ is trivial for all $x \in X$. Show that X is not locally path connected.

4. For any path f in a topological space X, let $[f]$ denote its homotype class with fixed end points. Consider the partially defined binary relation on the set of all path classes $[f]$, where $[f][g] = [f * g]$ if the end point of f equals the initial point of g. Here, $*$ denotes concatenation. Assuming this operation is well defined, show that it is associative, when that makes sense.

5. Let $p : (\widetilde{X}, \widetilde{x}) \to (X, x)$ be a covering projection.

   (a) Show that p is a local homeomorphism onto X, and its fibers are discrete.

   (b) If (Y, y) is a connected space then show that any pointed continuous function $f : (Y, y) \to (X, x)$ has at most one lift $\widetilde{f} : (Y, y) \to (\widetilde{X}, \widetilde{x})$.

   (c) Show that the group morphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x)$, induced by p, is injective. (In proving (c), you may assume the homotopy lifting theorem).